Quantum Phase Transition in Skyrmion Lattice

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We investigate the ground state of 2D electron gas in Quantum Hall regime at the filling factor slightly deviating from unity, that can be viewed as a sparse lattice of skyrmions. We have found that in the low density limit skyrmions are bound in pairs, those forming the actual lattice. We have shown that at further density increase the lattice undergoes a quantum phase transition, an analogue of superconducting phase transition in Josephson junction arrays.

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Skyrmion [1] has been a favorite model particle of field theorists for decades. Only recently it has been comprehended that the low-lying charged excitations of the Integer Quantum Hall ground state may be skyrmions([2], see [3] for review). This interesting development has prompted intensive experimental studies [4,5] that have proved these exotic particles to be real indeed. A skyrmion can be viewed as a topologically non-trivial distortion of the spins of *many* ground state electrons. That makes its spin much bigger than unity, although the skyrmion bears a unitary charge. The size of the skyrmions is controlled by a parameter $\tilde{g}$, the ratio of Zeeman energy to exchange energy per particle. The smaller the $\tilde{g}$ is, the bigger is the skyrmion.

If the filling factor slightly deviates from unity, the low-lying charged excitations must appear at the background of the Quantum Hall state to compensate for excess charge. Therefore, the deviation of the filling factor sets the skyrmion concentration and the many-skyrmion ground state can be easily made by changing either magnetic field or electron density of 2D gas. The two studies of many-skyrmion ground state at finite $\tilde{g}$ have been recently reported. [6,7] Despite the different methods applied and the different results obtained, it has been assumed in both studies that the skyrmions form a plain regular lattice. Also, in both studies no quantum effects have been taken into account.

Our results demonstrate that the formation and development of the multi-skyrmion state is more complicated and interesting than it has been assumed previously. We have shown that at low concentrations the skyrmions always appear in pairs. These pairs form a triangular Wigner lattice. With increasing concentration the distance between pairs decreases, and the interaction between the pairs strengthens. This leads to a quantum transition in skyrmion lattice. Below the transition point, the z-component of spin of each pair is well-defined. We call the variable that is canonically conjugated to z-component of spin the *phase*. Above the transition point, it is this phase that is well-defined.

We begin with the energy functional that describes spin textures in the limit of small $\tilde{g}$ (see e.g. [2,7]).

$$E = E_{stiff} + E_Z + E_{Coulomb}$$
$$= \int d^2r \left[\frac{\rho_s}{2}|\partial_\mu \mathbf{m}|^2 + \frac{|g|\mu_B B n_L}{2}(1+m_z)\right] + \frac{1}{2}\int d^2r \int d^2r' q(\mathbf{r})\frac{e^2}{\epsilon|\mathbf{r}-\mathbf{r}'|}q(\mathbf{r}'). \quad (1)$$

It contains three terms that describe respectively spin stiffness, Zeeman energy, and electrostatic energy. It depends on $\mathbf{m}(\mathbf{r})$, a unitary vector characterizing local spin density. We adopt the microscopic expression [8] for the spin stifness at filling factor 1: $\rho_s = (1/16\sqrt{2\pi})(e^2/\epsilon\lambda)$, λ being the magnetic length. In the Zeeman term, g stands for the electron g-factor, and μ_B is the Bohr magneton. In the electrostatic term, $q(\mathbf{r})$ presents the deviation of the electron density from the uniform background density $n_L = 1/2\pi\lambda^2$, q being related to the density of topological charge [2]:

$$q = -\frac{1}{8\pi}\epsilon_{\mu\nu}(\mathbf{m}[\partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}]). \quad (2)$$

The precise definition of $\tilde{g}$ reads $\tilde{g} \equiv g\mu_B B/E_{ex}$, $E_{ex} \equiv e^2/\epsilon\lambda$.

The stiffness term dominates the energy and must be minimized first of all. It has been shown by Belavin and Polyakov [9] that the textures minimizing the stiffness term are highly degenerate. They parametrized the spin density as follows:

$$W = \frac{m_x + im_y}{1-m_z}, \quad m_x + im_y = \frac{2W}{1+|W|^2}, \quad m_z = \frac{|W|^2-1}{|W|^2+1}. \quad (3)$$

It turns out that all the textures corresponding to any *analytical* function $W(z \equiv x+iy)$ with N poles possess the same energy $N\sqrt{\pi/32}E_{ex}$. Neither pole (skyrmion positions) nor pole residuals (skyrmion radii) are fixed. Those are determined by the interplay of the weaker interactions: Zeeman and electrostatic energy.

Let us start with a single skyrmion. [2] We take $W(z) = a/z$, a being the skyrmion radius to be evaluated. The Coulomb energy is given by $E_Q = (3\pi^2/64)e^2/\epsilon a$. The integral that gives Zeeman energy appears to diverge logarithmically at long distances from the core, since x, y spin components decrease very slowly with increasing distance. A slightly improved calculation shows that the divergency is to be cut at $r_s \simeq \lambda\tilde{g}^{-1/2}$, and Zeeman energy reads $E_Z = g\mu_B B(a^2/\lambda^2)\ln(r_s/a)$ provided $a \ll r_s$. The minimization of the total energy with respect to a yields $a \simeq \lambda(ln(1/\tilde{g})\tilde{g})^{-1/3}$, so that $a \ll r_s$ indeed.

Let us consider now *two* skyrmions with *opposite* phases,

$$W(z) = \frac{a}{z+R} - \frac{a}{z-R}, \tag{4}$$

the minus sign before the second term accounting for that phase shift. We see that the x, y spin components are effectively quenched at $|z| \gg R$, that lowers Zeeman energy. For well-separated skyrmions ($a \ll R$) the logarithmical divergency is cut at distances of the order of R and Zeeman energy reads $2g\mu_B B(a^2/\lambda^2)\ln(R/a)$. This results in an attractive force between skyrmions $\propto 1/R$. At long distances that always prevails Coulomb repulsion, so that the skyrmions are bound to form pairs. To find how they are exactly bound, we have studied the question numerically. Fig. 1 shows our results for the two skyrmion energy as function of their separation R, at optimal a. It appears that the minimum is achieved at zero separation, corresponding to $W(z) = b^2/z^2$. Two skyrmions merge as it sketched in Fig. 1. The optimal value of $|b|$ is $b_0 \simeq \lambda/\tilde{g}^{1/3}$. The skyrmion pair bears big spin, $S_0 \simeq 0.78/\tilde{g}^{2/3}$. We checked numerically that skyrmion attraction saturates, that is, there is no further bounding of pairs.

Therefore we have shown that the charged excitations with lowest energy are eventually skyrmion pairs bearing charge $\pm 2e$. We proceed now with analysing many-skyrmion ground state. [12] Let us compare now Zeeman and Coulomb contributions to the interaction between pairs. The latter appears to prevail provided the separation between pairs exceeds their equilibrium radius b_0. Therefore, the pairs behave very much like point-like charges and form a triangular Wigner lattice to minimize their Coulomb energy. Such an arrangement persist up to relatively high skyrmion densities $n_{sky}/n_{el} \simeq (b_0/\lambda)^{-2} \simeq \tilde{g}^{2/3}$ where they begin to overlap. Most probably, the overlap would depair skyrmions and one of the single-skyrmion lattices proposed would be realized at higher densities. In the present paper, we restrict our attention to lower densities where the pairs certainly form the triangular lattice.

To reveal the interesting physics that persists even in this limiting case, let us note that the two-skyrmion solution found is infinitely degenerate. All the spins can be rotated about z-axis by an arbitrary angle yielding the texture of the same energy. Or, alternatively, we can multiply $W(z)$ by an arbitrary phase factor $\exp(i\phi)$. We will call the variable parametrizing all the degenerate textures the *phase*. This degeneracy is not physical and arises from the fact that we treat the pair, which is a quantum particle, as a classical object.

Let us consider quantization of an isolated pair. We note that the shift in the phase space, $\phi \to \phi + \theta$, is just a rotation in spin space about z-axis being represented by the quantummechanical operator $\exp(-i\theta\hat{S}_z)$. Here $\hat{S}_z$ stands for the total spin of the skyrmion. Expanding near $\theta = 0$, we find that $S_z = -i\partial/\partial\phi$. That leads us to an important conslusion:

$$[\hat{\phi}, \hat{S}_z] = i; \tag{5}$$

the phase and the total spin of the pair are conjugate variables satisfying Heisenberg uncertainty relation. Owing to degeneracy with respect to phase, physical isolated skyrmion pairs do not have a certain phase but rather possess a certain integer z-component of spin. This picture has very much in common with Coulomb blockade in superconducting island [11,10], that does not have a certain superconducting phase but rather a certain integer charge.

Since the total spin of the pair is a function of its radius b, as well as its energy, the spin quantization leads to the energy quantization of pair states. The spectrum can be obtained from classical dependences $E(b)$ and $S_z(b)$. Near the minimum energy it reads

$$E = E_0 + E_c(S_z - S_0)^2, \quad E_c \equiv \frac{3}{4}\frac{\tilde{g}E_{ex}}{S_0}. \tag{6}$$

Here E_0, S_0 are optimal classical energy and spin respectively. Let us note that S_0 is a continious function of $\tilde{g}$, to be contrasted with integer S_z. If we change $\tilde{g}$, S_0 changes continuously, whereas the optimal S_z jumps between integer values. In the point of jump, energies of two states with different S_z match providing extra degeneracy.

This is reminiscent of Coulomb blockade phenomenon in Josephson arrays. [10] Josephson arrays consist of superconducting islands, the number of discrete charges in each island being canonically conjugate to its superconducting phase. The actual state of the array is determined by an interplay of charging energy and Josephson coupling. If charging energy dominates, the array is in insulating phase. There is a gap for charged excitations: one has to pay

charging energy in order to add or extract a discrete charge to/from the array. If the Josephson coupling increases, the quantum phase transition to the superconducting phase takes place. The long-range phase ordering appears, the gap vanishes.

We shall expect similar behaviour from skyrmion lattices, with discrete charge replaced by discrete S_z. At very low skyrmion concentrations the pairs can be regarded as isolated ones each having a well-defined spin and strongly fluctuating phase. The x, y components of the local spin density are undefined and have zero average, as it is sketched in Fig. 2a. There is a "spin gap" in this "inslulating" phase: one has to pay energy of the order of E_c to increase or decrease the spin of the system by unity. If we increase skyrmion concentration, the pairs come closer to each other. That increases coupling, that tries to fix the phases of neighboring pairs. Therefore we expect a quantum transition to "superconducting" phase. At the transition point, the "spin gap" vanishes to zero. Instead, long-range phase ordering appears. That is, x, y components of the local spin density become non-zero.

To give a quantitative description, we evaluate first the phase-dependent interaction between pairs. It appears that this interaction measures up E_c at interskyrmion distances much larger than screening radius r_s, $(r_s/\lambda)^2 = \sqrt{\pi}/(4\sqrt{2}\tilde{g})$ this is why we shall evaluate it for interpair distances r in the region $r \gg r_s, b$. We expand energy functional up to the terms quadratic in $s_{x,y}$ and minimize the resulting expression matching $s_{x,y}$ near the pair cores with asymptotics of $W(z)$. That yields

$$E_{int} = E_J \cos(\phi_1 - \phi_2), \quad E_J = g\mu_B B \frac{b_0^4}{r_s^4} \exp(-\frac{r}{r_s})(\frac{r_s}{\lambda^2}\sqrt{\pi r_s r/2}). \tag{7}$$

As a result, we have the following many-body Hamiltonian, which takes into account "charging" and Josephson energies:

$$\hat{H} = \sum_i E_c(-i\frac{\partial}{\partial \phi_i} - S_0)^2 + \frac{1}{2}{\sum_{ij}}' E_J \cos(\phi_i - \phi_j), \tag{8}$$

where E_c is given by Eq.(6), $E_J > 0$ is given by Eq.(7). Indexes label sites of triangular lattice, the prime restricts sum over j to the nearest neighbours of a certain site. Note that in our case the "Josephson" coupling has an antiferromagnetic sign, to be contrasted with Josepson arrays. In the limit $E_J \gg E_c$ the phases become well-defined and we can neglect "charging" term. The Hamiltonian reduces to energy functional. In the ground state, the lattice subdivides into three sublattices. [13] The phases of sublattices are rotated by $2\pi/3$ with respect to each other (Fig 2b).

We are interested in characterizing the transition. Since the exact solution of the Hamiltonian is beyond the reach, we find the transition point in Hartree approximation. We approximate the exact ground-state wave function by a product of site wave functions,

$$\Psi(\phi_1, ..., \phi_N) = \prod_{i=1}^{N} \Psi_i(\phi_i), \tag{9}$$

Ψ_i to be determined from the minimum energy principle. The site wave functions then satisfy the following mean-field Hamiltonian:

$$\hat{H}_{MF} = \sum_i E_c(-i\frac{\partial}{\partial \phi_i} - S_0)^2 + \frac{1}{4}E_J{\sum_{ij}}'[2n_j \exp(-i\phi_i) - n_i n_j^* + c.c.].$$

where we introduce the order parameter

$$n_i = \int_0^{2\pi} \exp(i\phi)|\Psi_i(\phi)|^2 d\phi$$

at each site. We assume that near transition point the order parameter has the same symmetry as for $E_J \gg E_c$. The non-trivial solution for the order parameter first appear at

$$E_J^0 = \frac{E_c}{3}\left\{1 - 4([S_0 + 1/2] - S_0)^2\right\}, \tag{10}$$

this equation determining the transition point. Making use of Eqs.(6, 7, 10) we find that the transition occurs at skyrmion density $n_{sky} \sim r_s^{-2}\ln^{-2}(1/\tilde{g})$. To give more detailed predictions, we plot the critical skyrmion density versus

$\tilde{g}$ in realistic range of this parameter. The curve exhibits sharp drops each time the energies of two spin states match, since this favours spin fluctuations and, consequently, facilitates phase ordering.

To observe the transition experimentally, one shall measure either spin gap or the local magnetization parallel to the plane, those quantities respectively vanish and appear above transition point. Both quantities are likely to be accessed in magnetic resonance measurement. The experiment shall be performed at temperature that is lower than the typical energy scale involved, E_c. Our estimations for $n_{el} = 10^{16} m^{-2}$, $\tilde{g} = 0.025$ give $E_c/k_B \simeq 1K$. The fine tuning of the filling factor in the range of 0.01 is required. Our model does not explicitly include disorder. However we expect that the quantum transition occurs even in disordered heterostructures, provided the skyrmions survive. The disorder may trap the skyrmions in random potential minima, thus preventing the formation of any regular lattice. Albeit it can not pin the skyrmion phase, that to be pinned only by interskyrmion interactions with increasing concentration. This is why we expect the quantum transition at $n_{sky}/n_{el} \simeq \tilde{g}$ even for disordered arrangement of skyrmions.

In conslusion, we have investigated many-skyrmion ground state at low skyrmion concentration. We have shown that the skyrmions are bound in pairs, those forming triangular Wigner lattice. The quantun phase transition occurs in this lattice at certain skyrmion concentration.

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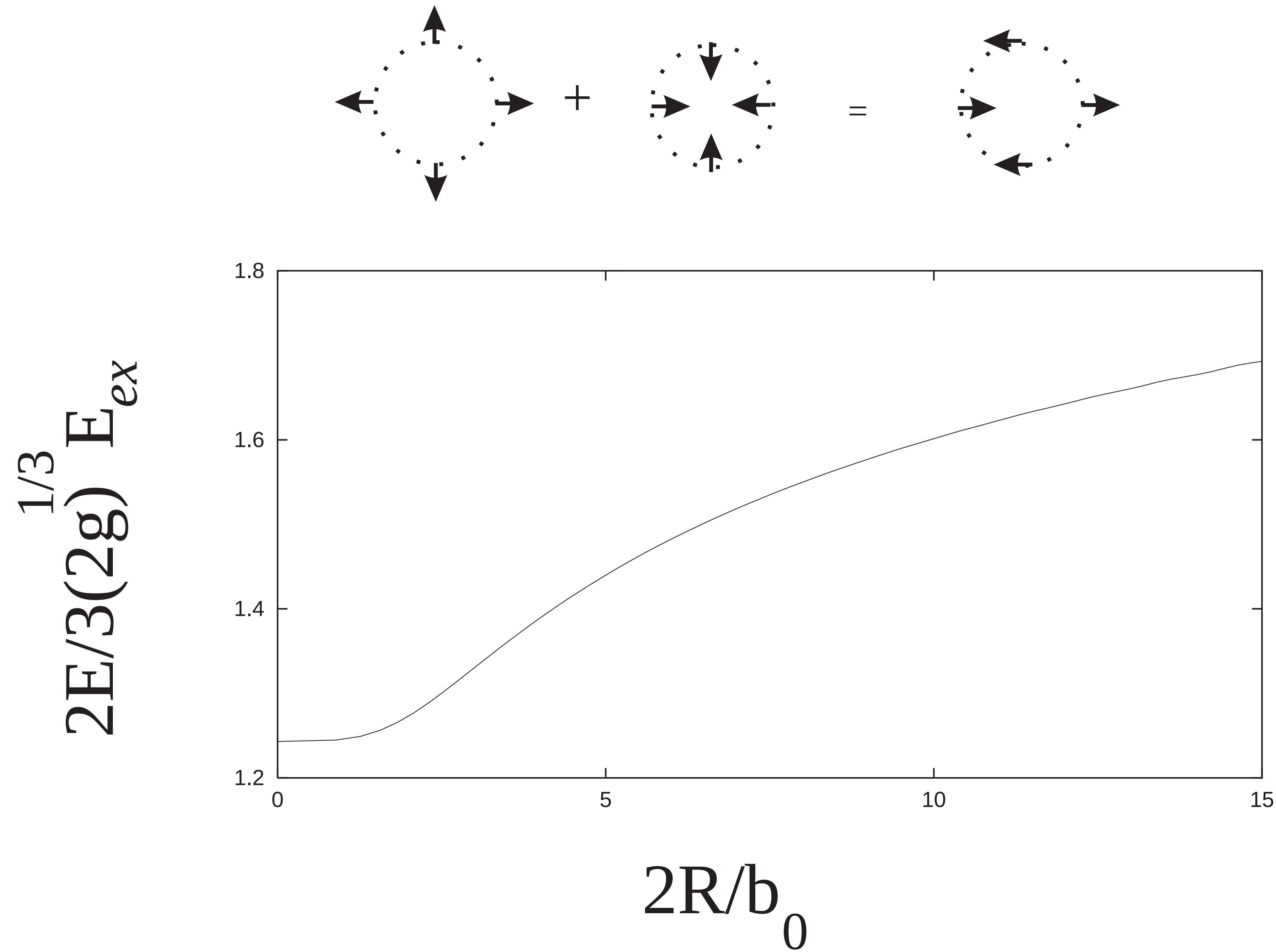
FIG. 1. Binding of two skyrmions. Two skyrmion energy as a function of their separation has a minimum at zero separation.

FIG. 2. Two phases of skyrmion lattice. In "insulating" phase, skyrmion phase strongly fluctuates so that *x,y* spin components have zero average. In "superconducting" phase, skyrmion phases are fixed and antiferromagnetically arranged

FIG. 3. The quantum phase transition. Critical skyrmion density is plotted in the region of realistic $\tilde{g}$. Above the line, the system is in "superconducting" state. Numbers denote the equilibrium spin of isolated skyrmion pairs at given $\tilde{g}$. The crtical line drops to zero any time the spin state changes.



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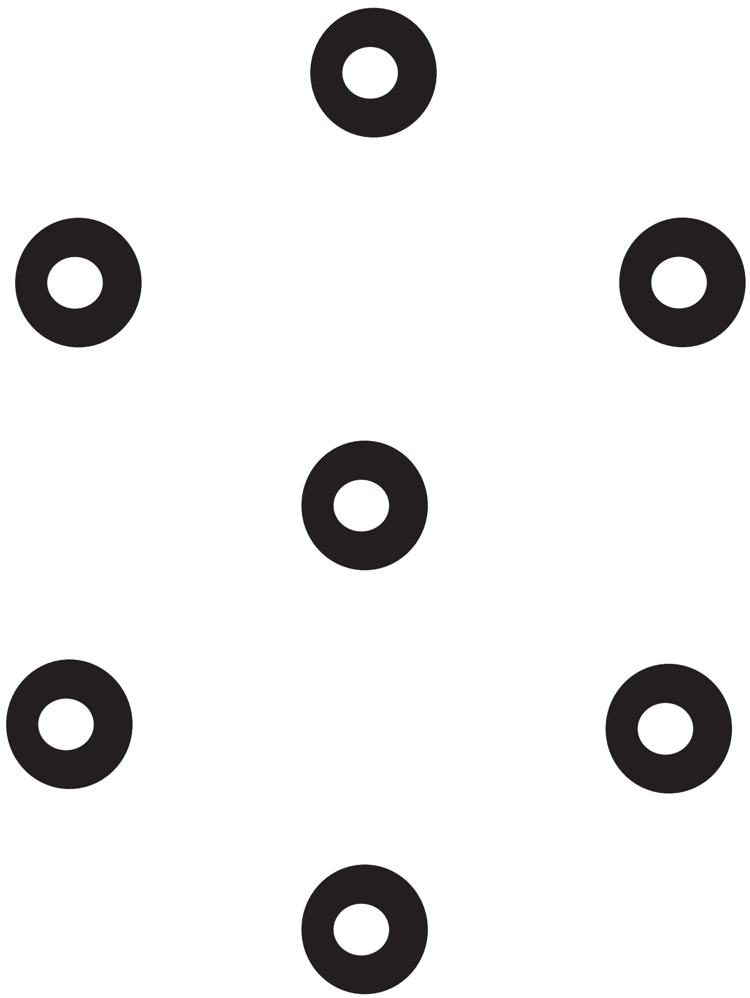
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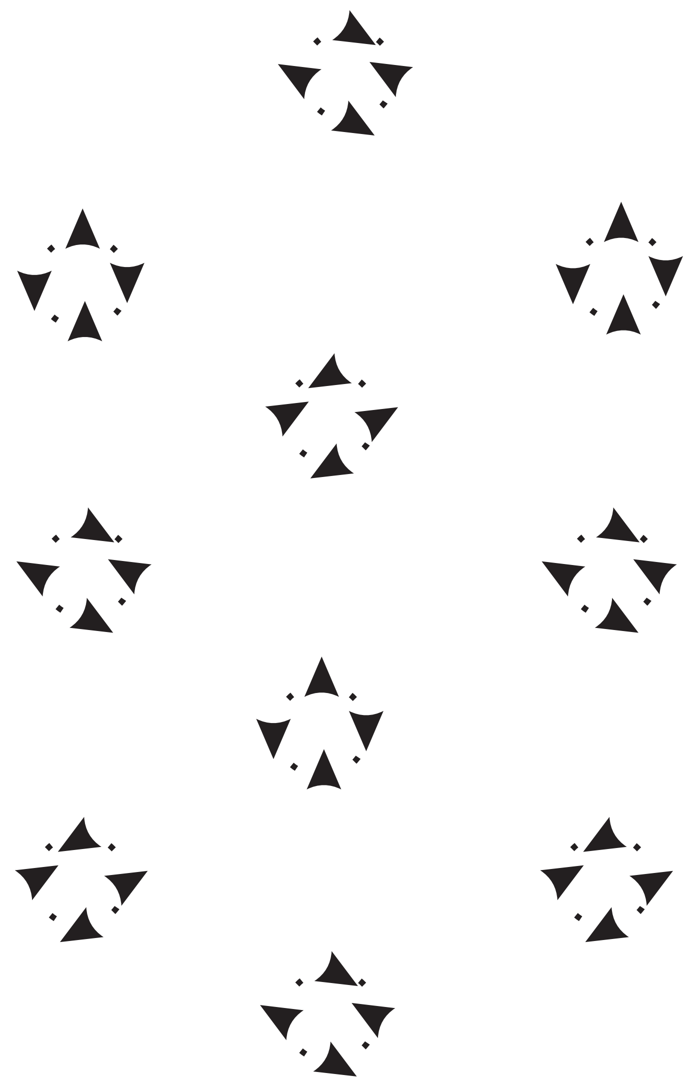
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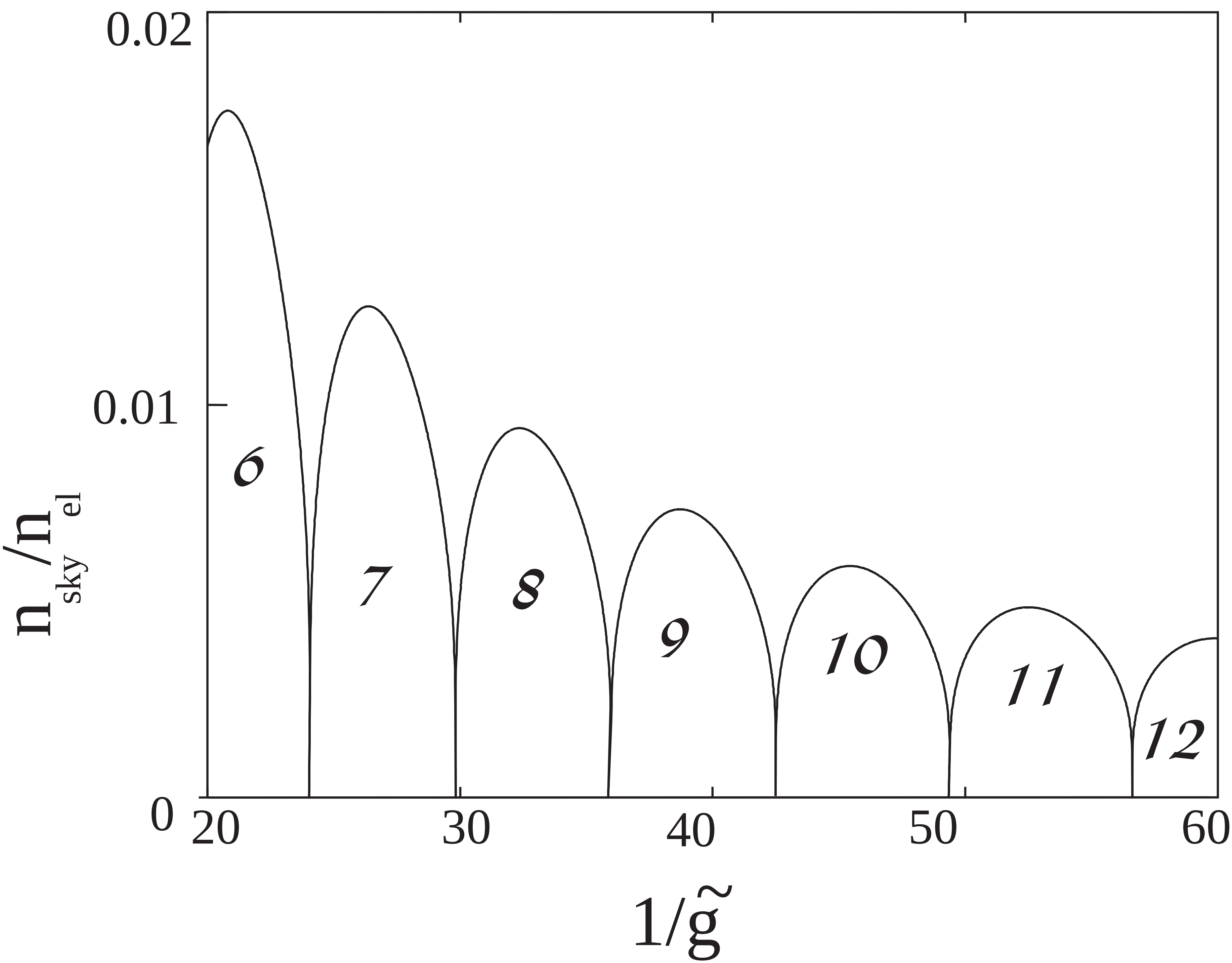
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